Vertex-transitive embeddings of graphs related to the degree-diameter and the degree-girth problems

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The Moore bounds and asymptotics for fixed k,ℓ and $d \to \infty$:

- $N(d,k) \le 1 + d + d(d-1) + \ldots + d(d-1)^{k-1} \sim d^k$
- ℓ odd: $n(d,\ell) \ge 1 + d + d(d-1) + \ldots + d(d-1)^{(\ell-3)/2} \sim d^{(\ell-1)/2}$
- ℓ even: $n(d,\ell) \ge 2[1 + (d-1) + \ldots + (d-1)^{(\ell-2)/2}] \sim 2d^{(\ell-2)/2}$

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Aim:

To explore possible vertex-transitive embeddings of these graphs.

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Theorem [Širáň and Tucker 2007]

Let Γ be a connected regular graph of valency at least three. Then, Γ has an orientably vertex-transitive embedding if and only if $Aut(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilizers.

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Corollary [Gardiner, Nedela, Širáň, Škoviera 1999]

A connected regular graph of valency ≥ 3 admits an orientably regular embedding (i.e., gives an orientably regular map) iff $Aut(\Gamma)$ contains a vertex-transitive subgroup with regular cyclic vertex stabilizers.

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Extensions to vertex-transitive maps admitting orientation-reversing automorphisms are also available but will not be discussed in this talk.

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Lifts $IP(q)$ of the dipole Δ, with $I = \{1, 2, \dots, q+1\}$, and voltages $\alpha(e_i) = z_i$ from a perfect difference set in Z_{q^2+q+1}; valency $q+1$, order $2(q^2+q+1)$;

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Proposition 1

The graph $IP(q)$ is a Cayley graph and hence admits a vertex-transitive embedding, but does not admit an orientably regular map if $q \notin \{2, 8\}$.

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Proposition 3

The graph $B'(q)$ admits an orientably regular embedding.

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AFLN1(q): Lifts of Δ^γ of the dipole Δ, with $I = F^\times$, and voltages on Δ: $\gamma(e_i) = (i, i) \in F^+ \times F^\times$ for $i \in I$, valency $q - 1$.

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Proposition 5

For prime n the graph $\Gamma(p^n)$ admits an orientably vertex-transitive embedding with cyclic stabilizer of order n.

The McKay-Miller-Širáň graphs

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Δ_q^* - dipole with q $u \to v$ edges e_x,
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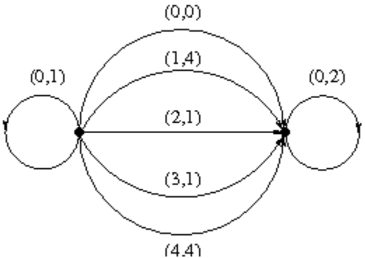

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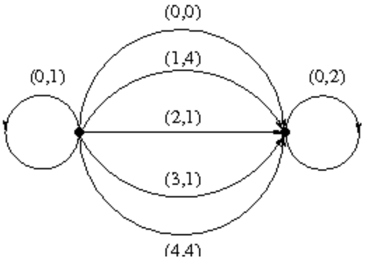

Proposition 6

$MMS(q)$ have no orientably vertex-transitive embeddings.

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Proposition 7

For every odd $d \ge 11$ and k such that $3 \le k \le (d+1)/2$ the graph $FMC(d, k)$ admits an orientably vertex-transitive embedding iff it is a Cayley graph. In particular, $FMC(d, k)$ has no such embedding unless $k = (d \pm 1)/2$ or $(k, d) \in \{(5, 21), (4, 19), (3, 2q-1)\}$, q a prime power.

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THANK YOU!